MATH 2050A: Mathematical Analysis I (2017 1st term)

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1 Compact Sets in $\mathbb{R}$

Throughout this section, let (x_n) be a sequence in $\mathbb{R}$. Recall that a subsequence $(x_{n_k})_{k=1}^{\infty}$ of (x_n) means that $(n_k)_{k=1}^{\infty}$ is a sequence of positive integers satisfying $n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$, that is, such sequence (n_k) can be viewed as a strictly increasing function $\mathbf{n} : k \in \{1, 2, ..\} \mapsto n_k \in \{1, 2, ...\}$.
In this case, note that for each positive integer N, there is $K \in \mathbb{N}$ such that $n_K \geq N$ and thus we have $n_k \geq N$ for all $k \geq K$.

Let us first recall the following two important theorems in real line.

Theorem 1.1 Nested Intervals Theorem *Let* $(I_n := [a_n, b_n])$ *be a sequence of closed and bounded intervals. Suppose that it satisfies the following conditions.*

(i) : $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$.

(ii) : $\lim_n (b_n - a_n) = 0$.

Then there is a unique real number ξ *such that* $\bigcap_{n=1}^{\infty} I_n = \{\xi\}$.

Proof: See [1, Theorem 2.5.2, Theorem 2.5.3]. □

Theorem 1.2 (Bolzano-Weierstrass Theorem) *Every bounded sequence in* $\mathbb{R}$ *has a convergent subsequence.*

Proof: See [1, Theorem 3.4.8]. □

Definition 1.3 A subset A of $\mathbb{R}$ is said to be *compact* (more precise, *sequentially compact*) if every sequence in A has a convergent subsequence with the limit in A.

We are now going to characterize the compact subsets of $\mathbb{R}$. The following is an important notation in mathematics.

Definition 1.4 A subset A is said to be *closed* in $\mathbb{R}$ if it satisfies the condition:

if (x_n) *is a sequence in* A *and* $\lim x_n$ *exists, then* $\lim x_n \in A$.

Example 1.5 (i) $\{a\}; [a, b]; [0, 1] \cup \{2\}$; $\mathbb{N}$; the empty set $\emptyset$ and $\mathbb{R}$ all are closed subsets of $\mathbb{R}$.

(ii) (a, b) and $\mathbb{Q}$ are not closed.

The following Proposition is one of the basic properties of a closed subset which can be directly shown by the definition. So, the proof is omitted here.

Proposition 1.6 *Let A be a subset of $\mathbb{R}$. The following statements are equivalent.*

(i) A is closed.

(ii) For each element $x \in \mathbb{R} \setminus A$, there is $\delta_x > 0$ such that $(x - \delta_x, x + \delta_x) \cap A = \emptyset$.

The following is an important characterization of a compact set in $\mathbb{R}$. **Warning:** this result is not true for the so-called *metric spaces* in general.

Theorem 1.7 *Let A be a closed subset of $\mathbb{R}$. Then the following statements are equivalent.*

(i) A is compact.

(ii) A is closed and bounded.

Proof: It is clear that the result follows if $A = \emptyset$. So, we assume that A is non-empty.
For showing $(i) \Rightarrow (ii)$, assume that A is compact.
We first claim that A is closed. Let (x_n) be a sequence in A. Then by the compactness of A, there is a convergent subsequence (x_{n_k}) of (x_n) with $\lim_k x_{n_k} \in A$. So, if (x_n) is convergent, then $\lim_n x_n = \lim_k x_{n_k} \in A$. Therefore, A is closed.
Next, we are going to show the boundedness of A. Suppose that A is not bounded. Fix an element $x_1 \in A$. Since A is not bounded, we can find an element $x_2 \in A$ such that $|x_2 - x_1| > 1$. Similarly, there is an element $x_3 \in A$ such that $|x_3 - x_k| > 1$ for $k = 1, 2$. To repeat the same step, we can obtain a sequence (x_n) in A such that $|x_n - x_m| > 1$ for $m \neq n$. From this, we see that the sequence (x_n) does not have a convergent subsequence. In fact, if (x_n) has a convergent subsequence (x_{n_k}). Put $L := \lim_k x_{n_k}$. Then we can find a pair of sufficient large positive integers p and q with $p \neq q$ such that $|x_{n_p} - L| < 1/2$ and $|x_{n_q} - L| < 1/2$. This implies that $|x_{n_p} - x_{n_q}| < 1$. It leads to a contradiction because $|x_{n_p} - x_{n_q}| > 1$ by the choice of the sequence (x_n). Thus, A is bounded.
It remains to show $(ii) \Rightarrow (i)$. Suppose that A is closed and bounded.
Let (x_n) be a sequence in A. Thus, (x_n). Then the Bolzano-Weierstrass Theorem assures that there is a convergent subsequence (x_{n_k}). Then by the closeness of A, $\lim_k x_{n_k} \in A$. Thus A is compact.
The proof is finished.
□

2 Appendix: Compact sets in $\mathbb{R}$, Part 2

For convenience, we call a collection of open intervals $\{J_\alpha : \alpha \in \Lambda\}$ an *open intervals cover* of a given subset A of $\mathbb{R}$, where Λ is an arbitrary non-empty index set, if each J_α is an open interval (not necessary bounded) and

$$A \subseteq \bigcup_{\alpha \in \Lambda} J_\alpha.$$

Theorem 2.1 Heine-Borel Theorem: *Any closed and bounded interval $[a,b]$ satisfies the following condition:*

(HB) Given any open intervals cover $\{J_\alpha\}_{\alpha\in\Lambda}$ of $[a,b]$, we can find finitely many $J_{\alpha_1},..,J_{\alpha_N}$ such that $[a,b] \subseteq J_{\alpha_1} \cup \cdots \cup J_{\alpha_N}$

Proof: Suppose that $[a,b]$ does not satisfy the above Condition (HB). Then there is an open intervals cover $\{J_\alpha\}_{\alpha\in\Lambda}$ of $[a,b]$ but it it has no finite sub-cover. Let $I_1 := [a_1, b_1] = [a,b]$ and m_1 the mid-point of $[a_1,b_1]$. Then by the assumption, $[a_1,m_1]$ or $[m_1,b_1]$ cannot be covered by finitely many J_α's. We may assume that $[a_1,m_1]$ cannot be covered by finitely many J_α's. Put $I_2 := [a_2,b_2] = [a_1,m_1]$. To repeat the same steps, we can obtain a sequence of closed and bounded intervals $I_n = [a_n,b_n]$ with the following properties:

(a) $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots\cdots$;

(b) $\lim_n (b_n - a_n) = 0$;

(c) each I_n cannot be covered by finitely many J_α's.

Then by the Nested Intervals Theorem, there is an element $\xi \in \bigcap_n I_n$ such that $\lim_n a_n = \lim_n b_n = \xi$. In particular, we have $a = a_1 \le \xi \le b_1 = b$. So, there is $\alpha_0 \in \Lambda$ such that $\xi \in J_{\alpha_0}$. Since J_{α_0} is open, there is $\varepsilon > 0$ such that $(\xi - \varepsilon, \xi + \varepsilon) \subseteq J_{\alpha_0}$. On the other hand, there is $N \in \mathbb{N}$ such that a_N and b_N in $(\xi - \varepsilon, \xi + \varepsilon)$ because $\lim_n a_n = \lim_n b_n = \xi$. Thus we have $I_N = [a_N, b_N] \subseteq (\xi - \varepsilon, \xi + \varepsilon) \subseteq J_{\alpha_0}$. It contradicts to the Property (c) above. The proof is finished.
□

Remark 2.2 The assumption of the closeness and boundedness of an interval in Heine-Borel Theorem is essential.
For example, notice that $\{J_n := (1/n, 1) : n = 1, 2...\}$ is an open interval covers of $(0,1)$ but you cannot find finitely many J_n's to cover the open interval $(0,1)$.

The following is a very important feature of a compact set.

Theorem 2.3 *Let A be a subset of $\mathbb{R}$. Then the following statements are equivalent.*

(i) For any open intervals cover $\{J_\alpha\}_{\alpha\in\Lambda}$ of A, we can find finitely many $J_{\alpha_1},..,J_{\alpha_N}$ such that $A \subseteq J_{\alpha_1} \cup \cdots \cup J_{\alpha_N}$.

(ii) A is compact.

(iii) A is closed and bounded.

Proof: The result will be shown by the following path

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i).$$

For $(i) \Rightarrow (ii)$, assume that the condition (i) holds but A is not compact. Then there is a sequence (x_n) in A such that (x_n) has no subsequent which has the limit in A. Put $X = \{x_n : n = 1, 2, ...\}$. Then X is infinite. Also, for each element $a \in A$, there is $\delta_a > 0$ such that $J_a := (a-\delta_a, a+\delta_a) \cap X$ is finite. Indeed, if there is an element $a \in A$ such that $(a-\delta, a+\delta)\cap A$ is infinite for all $\delta > 0$, then (x_n) has a convergent subsequence with the limit a. On the other hand, we have $A \subseteq \bigcup_{a\in A} J_a$. Then by the compactness of A, we can find finitely many $a_1, ..., a_N$ such that $A \subseteq J_{a_1} \cup \cdots \cup J_{a_N}$. So we have $X \subseteq J_{a_1} \cup \cdots \cup J_{a_N}$. Then by the choice of J_a's, X must be finite. This leads to a contradiction. Therefore, A must be compact.
The implication $(ii) \Rightarrow (iii)$ follows from Theorem 1.7 at once.
It remains to show $(iii) \Rightarrow (i)$. Suppose that A is closed and bounded. Then we can find a closed and bounded interval $[a, b]$ such that $A \subseteq [a, b]$. Now let $\{J_\alpha\}_{\alpha\in\Lambda}$ be an open intervals cover of A. Notice that for each element $x \in [a, b] \setminus A$, there is $\delta_x > 0$ such that $(x - \delta_x, x + \delta_x) \cap A = \emptyset$ since A is closed by using Proposition 6.4. If we put $I_x = (x - \delta_x, x + \delta_x)$ for $x \in [a, b] \setminus A$, then we have

$$[a, b] \subseteq \bigcup_{\alpha\in\Lambda} J_\alpha \cup \bigcup_{x\in[a,b]\setminus A} I_x.$$

Using the Heine-Borel Theorem 2.1, we can find finitely many J_α's and I_x's, say $J_{\alpha_1}, ..., J_{\alpha_N}$ and $I_{x_1}, ..., I_{x_K}$, such that $A \subseteq [a, b] \subseteq J_{\alpha_1} \cup \cdots \cup J_{\alpha_N} \cup I_{x_1} \cup \cdots \cup I_{x_K}$. Note that $I_x \cap A = \emptyset$ for each $x \in [a, b] \setminus A$ by the choice of I_x. Therefore, we have $A \subseteq J_{\alpha_1} \cup \cdots \cup J_{\alpha_N}$ and hence A is compact.
The proof is finished. □

Remark 2.4 In fact, the condition in Theorem 2.3(i) is the usual definition of a *compact set* for a general topological space. More precise, if a set A satisfies the Definition 1.4, then A is said to be *sequentially compact.* Theorem 2.3 tells us that the notation of the compactness and the sequentially compactness are the same as in the case of a subset of $\mathbb{R}$. However, these two notation are different for a general topological space.

Strongly recommended: take the courses: MATH 3060; MATH3070 for the next step.

3 Continuous functions defined on compact sets

Throughout this section, let A be a non-empty subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$ a function defined on A.

Proposition 3.1 *Let f be a continuous function defined on a compact subset A of $\mathbb{R}$. Then $f(A)$ is a compact subset of $\mathbb{R}$.*

Proof: **Method I**: By using Theorem 2.3 $(i) \Leftrightarrow (iii)$, it suffices to show that $f(A)$ is a closed bounded subset of $\mathbb{R}$.
Claim 1: $f(A)$ is bounded.
Suppose not. Then for each positive integer n, there is an element $x_n \in A$ such that $|f(x_n)| > n$.

Since A is compact, there is a convergent subsequence (x_{n_k}) with $a := \lim_k x_{n_k} \in A$. This gives $\lim_k f(x_{n_k}) = f(a)$ because f is continuous on a and hence, $(f(x_{n_k}))$ is a bounded sequence. This leads to a contradiction to the choice of (x_n) which satisfies $|f(x_{n_k})| > n_k$ for all $k = 1, 2....$
Claim 2: $f(A)$ is a closed subset of $\mathbb{R}$, that is, $y \in f(A)$ whenever, a sequence (x_n) in A satisfying $\lim_n f(x_n) = y$.
In fact, there is a convergent subsequence (x_k) with $z := \lim_k x_k \in A$ by using the compactness of A again. This gives $y = \lim_k f(x_{n_k}) = f(z) \in f(A)$ as desired since f is continuous on A.
Method II: Alternatively, we are going to use Theorem 2.3 $(i) \Leftrightarrow (ii)$.
Let $\{J_i\}_{i\in I}$ be an open interval covers of $f(A)$. We may assume $J_i \cap f(A) \neq \emptyset$ for each $i \in I$. Notice that since J_i is an open interval and f is continuous, we see that if $f(x) \in J_i$, then we can find $\delta_x > 0$ such that $f(z) \in J_i$ whenever $z \in A$ with $|z - x| < \delta_x$. Notice that we have $A \subseteq \bigcup_{x\in A} J_x$, where $V_x := (x - \delta_x, x + \delta_x)$ and hence, $\{V_x : x \in A\}$ forms an open intervals cover of A. By using the equivalence $(i) \Leftrightarrow (ii)$ in Theorem 2.3, we can find finitely many $x_1, ..., x_n$ in A such that $A \subseteq V_{x_1} \cup \cdots \cup V_{x_n}$. For each $k = 1, .., n$, then $f(x_k) \in J_{i_k}$ for some $i_k \in I$. Now if $x \in A$, then $x \in V_{x_k}$ for some $k = 1, ..., n$. This gives $f(x) \in J_{i_k}$ and thus, $f(A) \subseteq J_{i_1} \cup \cdots \cup J_{i_n}$. The proof is finished. □

Corollary 3.2 *If $f : A \to \mathbb{R}$ is a continuous injection and A is compact, then the inverse map $f^{-1} : f(A) \to A$ is also continuous.*

Proof: Let $B = f(A)$ and $g = f^{-1} : B \to A$. Suppose that g is not continuous at some $b \in B$. Put $a = g(b) \in A$. Then there are $\eta > 0$ and a sequence (y_n) in B such that $\lim y_n = b$ but $|g(y_n) - g(b)| \geq \eta$ for all n. Let $x_n := g(y_n) \in A$. So, by the compactness of A, there is a convergent subsequence (x_{n_k}) of (x_n) such that $\lim_k x_{n_k} \in A$. Let $a' = \lim_k x_{n_k}$. Then we have $f(a') = \lim_k f(x_{n_k}) = \lim_k y_{n_k} = b$. On the other hand, since $|g(y_n) - g(b)| \geq \eta$ for all n, we see that

$$|x_{n_k} - a| = |g(y_{n_k}) - g(b)| \geq \eta > 0$$

for all k and hence $|a' - a| > 0$. This implies that $a \neq a'$ but $f(a') = b = f(a)$. It contradicts to f being injective.
The proof is finished. □

Remark 3.3 The assumption of the compactness in the last assertion of Proposition 3.2 is essential. For example, consider $A = [0, 1) \cup [2, 3]$ and define $f : A \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \in [0, 1) \\ x - 1 & \text{if } x \in [2, 3]. \end{cases}$$

Then $f(A) = [0, 2]$ and f is a continuous bijection from A onto $[0, 2]$ but $f^{-1} : [0, 2] \to A$ is not continuous at $y = 1$.

Example 3.4 By Proposition 3.2, it is impossible to find a continuous surjection from $[0, 1]$ onto $(0, 1)$ since $[0, 1]$ is compact but $(0, 1)$ is not. Thus $[0, 1]$ is not homeomorphic to $(0, 1)$.

Proposition 3.5 *Suppose that f is continuous on A. If A is compact, then there are points c and b in A such that*

$$f(c) = \max\{f(x) : x \in A\} \text{ and } f(b) = \min\{f(x) : x \in A\}.$$

Proof: By considering the function $-f$ on A, it needs to show that $f(c) = \max\{f(x) : x \in A\}$ for some $c \in A$.

Method I:

We first claim that f is bounded on A, that is, there is $M > 0$ such that $|f(x)| \leq M$ for all $x \in A$. Suppose not. Then for each $n \in \mathbb{N}$, we can find $a_n \in A$ such that $|f(a_n)| > n$. Recall that A is compact if and only if it is closed and bounded (see Theorem **??**). So, (a_n) is a bounded sequence in A. Then by the Bolzano-Weierstrass Theorem, there is a convergent subsequence (a_{n_k}) of (a_n). Put $a = \lim_k a_{n_k}$. Since A is closed and f is continuous, $a \in A$, from this, it follows that $f(a) = \lim_k f(a_{n_k})$. It is absurd because $n_k < |f(a_{n_k})| \to |f(a)|$ for all k and $n_k \to \infty$. So f must be bounded. So $L := \sup\{f(x) : x \in A\}$ must exist by the Axiom of Completeness.

It remains to show that there is a point $c \in A$ such that $f(c) = L$. In fact, by the definition of supremum, there is a sequence (x_n) in A such that $\lim_n f(x_n) = L$. Then by the Bolzano-Weierstrass Theorem again, there is a convergent subsequence (x_{n_k}) of (x_n) with $\lim_k x_{n_k} \in A$. If we put $c := \lim_k x_{n_k} \in A$, then $f(c) = \lim_k f(x_{n_k}) = L$ as desired. The proof is finished.

Method II:

We first claim that f is bounded above. Notice that for each $x \in A$, there is $\delta_x > 0$ such that $f(y) < f(x) + 1$ whenever $y \in A$ with $|x - y| < \delta_x$ since f is continuous on A. Now if we put $J_x := (x - \delta_x, x + \delta_x)$ for each $x \in A$, then $A \subseteq \bigcup_{x \in A} J_x$. So, by the compactness of A, we can find finitely many $x_1, ..., x_N$ in A such that $A \subseteq J_{x_1} \cup \cdots \cup J_{x_N}$ and it follows that for each $x \in A$, we have $f(x) < 1 + f(x_k)$ for some $k = 1, ..., N$. Now if we put $M := \max\{1 + f(x_1), ..., 1 + f(x_N)\}$, then f is bounded above by M on A.

Put $L := \sup\{f(x) : x \in A\}$. It remains to show that there is an element $c \in A$ such that $f(c) = L$. Suppose not. Notice that since $f(x) \leq L$ for all $x \in A$, we have $f(x) < L$ for all $x \in A$ under this assumption. Therefore, by the continuity of f, for each $x \in A$, there are $\varepsilon_x > 0$ and $\eta_x > 0$ such that $f(y) < f(x) + \varepsilon_x < L$ whenever $y \in A$ with $|y - x| < \delta_x$. Put $I_x := (x - \eta_x, x + \eta_x)$. Then $A \subseteq \bigcup_{x \in A} I_x$. By the compactness of A again, A can be covered by finitely many $I_{x_1}, ..., I_{x_N}$. If we let $L' := \max\{f(x_1) + \varepsilon_{x_1}, ..., f(x_N) + \varepsilon_{x_N}\}$, then $f(x) < L' < L$ for all $x \in A$. It contradicts to L being the least upper bound for the set $\{f(x) : x \in A\}$. The proof is complete. □

Definition 3.6 We say that a function f is *upper semi-continuous (resp. lower semi-continuous)* on A if for each element $z \in A$ and for any $\varepsilon > 0$, there is $\delta > 0$ such that $f(x) < f(z) + \varepsilon$ (resp. $f(z) - \varepsilon < f(x)$) whenever $x \in A$ with $|x - z| < \delta$.

Remark 3.7 (i) It is clear that a function is continuous if and only if it is upper semi-continuous and lower semi-continuous. However, an upper semi-continuous function need not be continuous. For example, define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

(ii) From the **Method II** above, we see that if f is upper semi-continuous (resp. lower semi-continuous) on a compact set A, then the function f attains the supremum (resp. infimum) on A.

4 Uniform Continuous Functions

Definition 4.1 A function $f : A \to \mathbb{R}$ is said to be uniformly continuous on A if for any $\varepsilon > 0$, there is $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in A$ with $|x - y| < \delta$.

Remark 4.2 It is clear that if f is uniformly continuous on A, then it must be continuous on A. However, the converse does not hold. For example, consider the function $f : (0, 1] \to \mathbb{R}$ defined by $f(x) := 1/x$. Then f is continuous on $(0, 1]$ but it is not uniformly continuous on $(0, 1]$. Notice that f is not uniformly continuous on A means that

there is $\varepsilon > 0$ such that for any $\delta > 0$, there are $x, y \in A$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$.

Notice that $1/x \to \infty$ as $x \to 0+$. So if we let $\varepsilon = 1$, then for any $\delta > 0$, we choose $n \in \mathbb{N}$ such that $1/n < \delta$ and thus we have $|1/2n - 1/n| = 1/2n < \delta$ but $|f(1/n) - f(1/2n)| = n > 1 = \varepsilon$. Therefore, f is not uniformly continuous on $(0, 1]$.

Example 4.3 Let $0 < a < 1$. Define $f(x) = 1/x$ for $x \in [a, 1]$. Then f is uniformly continuous on $[a, 1]$. In fact for $x, y \in [a, 1]$, we have

$$|f(x) - f(y)| = |\frac{1}{x} - \frac{1}{y}| = \frac{|x - y|}{xy} \leq \frac{|x - y|}{a^2}.$$

So for any $\varepsilon > 0$, we can take $0 < \delta < a^2\varepsilon$. Thus if $x, y \in [a, 1]$ with $|x - y| < \delta$, then we have $|f(x) - f(y)| < \varepsilon$ and hence f is uniformly continuous on $[a, 1]$.

Proposition 4.4 *If f is continuous on a compact set A, then f is uniformly continuous on A.*

Proof: **Compactness argument**:
Let $\varepsilon > 0$. Since f is continuous on A, then for each $x \in A$, there is $\delta_x > 0$, such that $|f(y) - f(x)| < \varepsilon$ whenever $y \in A$ with $|y - x| < \delta_x$. Now for each $x \in A$, set $J_x = (x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2})$. Then $A \subseteq \bigcup_{x \in A} J_x$. By the compactness of A, there are finitely many $x_1, ..., x_N \in A$ such that $A \subseteq J_{x_1} \cup \cdots \cup J_{x_N}$. Now take $0 < \delta < \min(\frac{\delta_{x_1}}{2}, ..., \frac{\delta_{x_N}}{2})$. Now for $x, y \in A$ with $|x - y| < \delta$, then $x \in I_{x_k}$ for some $k = 1, .., N$, from this it follows that $|x - x_k| < \frac{\delta_{x_k}}{2}$ and $|y - x_k| \leq |y - x| + |x - x_k| \leq \frac{\delta_{x_k}}{2} + \frac{\delta_{x_k}}{2} = \delta_{x_k}$. So for the choice of δ_{x_k}, we have $|f(y) - f(x_k)| < \varepsilon$ and $|f(x) - f(x_k)| < \varepsilon$. Thus we have shown that $|f(x) - f(y)| < 2\varepsilon$ whenever $x, y \in A$ with $|x - y| < \delta$. The proof is finished.
Sequentially compactness argument:
Suppose that f is not uniformly continuous on A. Then there is $\varepsilon > 0$ such that for each $n = 1, 2, ..,$ we can find x_n and y_n in A with $|x_n - y_n| < 1/n$ but $|f(x_n) - f(y_n)| \geq \varepsilon$. Notice that by the sequentially compactness of A, (x_n) has a convergent subsequence (x_{n_k}) with $a := \lim_k x_{n_k} \in A$. Now applying sequentially compactness of A for the sequence (y_{n_k}), then (y_{n_k}) contains a convergent subsequence $(y_{n_{k_j}})$ such that $b := \lim_j y_{n_{k_j}} \in A$. On the other hand, we also have $\lim_j x_{n_{k_j}} = a$. Since $|x_{n_{k_j}} - y_{n_{k_j}}| < 1/n_{k_j}$ for all j, we see that $a = b$. This implies that $\lim_j f(x_{n_{k_j}}) = f(a) = f(b) = \lim_j f(y_{n_{k_j}})$. This leads to a contradiction since we always have $|f(x_{n_{k_j}}) - f(y_{n_{k_j}})| \geq \varepsilon > 0$ for all j by the choice of x_n and y_n above. The proof is finished. □

Proposition 4.5 *Let f be a continuous function defined on a bounded subset A of $\mathbb{R}$. Then the following statements are equivalent.*

(i): f is uniformly continuous on A.

(ii): There is a unique continuous function F defined on the closure $\overline{A}$ such that $F(x) = f(x)$ for all $x \in A$.

Proof: Notice that since A is bounded then so is $\overline{A}$. This implies that $\overline{A}$ is compact. The Part $(ii) \Rightarrow (i)$ follows Proposition 4.4 at once.
The proof of Part $(i) \Rightarrow (ii)$ is divided by the following assertions. Assume that f is uniformly continuous on A.
Claim 1. If (x_n) is a sequence in A and $\lim x_n$ exists, then $\lim f(x_n)$ exists.
It needs to show that $(f(x_n))$ is a Cauchy sequence. Indeed, let $\varepsilon > 0$. Then by the uniform continuity of f on A, there is $\delta > 0$ such that $|f(x)-f(y)| < \varepsilon$ whenever $x, y \in A$ with $|x-y| < \delta$. Notice that (x_n) is a Cauchy sequence since it is convergent. Thus, there is a positive integer N such that $|x_m - x_n| < \delta$ for all $m, n \geq N$. This implies that $|f(x_m) - f(x_n)| < \varepsilon$ for all $m, n \geq N$ and hence, **Claim 1** follows.
Claim 2. If (x_n) and (y_n) both are convergent sequences in A and $\lim x_n = \lim y_n$, then $\lim f(x_n) = \lim f(y_n)$.
By **Claim 1**, $L := \lim f(x_n)$ and $L' = \lim f(y_n)$ both exist. For any $\varepsilon > 0$, let $\delta > 0$ be found as in **Claim 1**. Since $\lim x_n = \lim y_n$, there is $N \in \mathbb{N}$ such that $|x_n - y_n| < \delta$ for all $n \geq N$ and hence, we have $|f(x_n) - f(y_n)| < \varepsilon$ for all $n \geq N$. Taking $n \to \infty$, we see that $|L - L'| \leq \varepsilon$ for all $\varepsilon > 0$. So $L = L'$. **Claim 2** follows.
Recall that an element $x \in \overline{A}$ if and only if there is a sequence (x_n) in A converging to x.
Now for each $x \in \overline{A}$, we define

$$F(x) := \lim f(x_n)$$

if (x_n) is a sequence in A with $\lim x_n = x$. It follows from **Claim 1** and **Claim 2** that F is a well defined function defined on $\overline{A}$ and $F(x) = f(x)$ for all $x \in A$.
So, it remains to show that F is continuous. Then F is a continuous extension of f to $\overline{A}$ as desired.
Now suppose that F is not continuous at some point $z \in \overline{A}$. Then there is $\varepsilon > 0$ such that for any $\delta > 0$, there is $x \in \overline{A}$ satisfying $|x - z| < \delta$ but $|F(x) - F(z)| \geq \varepsilon$. Notice that for any $\delta > 0$ and if $|x - z| < \delta$ for some $x \in \overline{A}$, then we can choose a sequence (x_i) in A such that $\lim x_i = x$. Therefore, we have $|x_i - z| < \delta$ and $|f(x_i) - F(z)| \geq \varepsilon/2$ for any i large enough. Therefore, for any $\delta > 0$, we can find an element $x \in A$ with $|x-z| < \delta$ but $|f(x)-F(z)| \geq \varepsilon/2$. Now consider $\delta = 1/n$ for $n = 1, 2....$ This yields a sequence (x_n) in A which converges to z but $|f(x_n) - F(z)| \geq \varepsilon/2$ for all n. However, we have $\lim f(x_n) = F(z)$ by the definition of F which leads to a contradiction. Thus F is continuous on $\overline{A}$.
Finally the uniqueness of such continuous extension is clear.
The proof is finished. □

Example 4.6 By using Proposition 4.5, the function $f(x) := \sin \frac{1}{x}$ defined on $(0, 1]$ cannot be continuously extended to the set $[0, 1]$.

Definition 4.7 Let A be a non-empty subset of $\mathbb{R}$. A function $f : A \to \mathbb{R}$ is called a Lipschitz if there is a constant $C > 0$ such that $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in A$. In this case. Furthermore, if we can find such $0 < C < 1$, then we call f a contraction.

It is clear that we have the following property.

Proposition 4.8 *Every Lipschitz function is uniformly continuous on its domain.*

Example 4.9 (i) : The sine function $f(x) = \sin x$ is a Lipschitz function on $\mathbb{R}$ since we always have $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$ (by using the equation $\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$ and the fact $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$.)

(ii) : Define a function f on $[0, 1]$ by $f(x) = x\sin(1/x)$ for $x \in (0, 1]$ and $f(0) = 0$. Then f is continuous on $[0, 1]$ and thus f is uniformly continuous on $[0, 1]$. But notice that f is not a Lipschitz function. In fact, for any $C > 0$, if we consider $x_n = \frac{1}{2n\pi+(\pi/2)}$ and $y_n = \frac{1}{2n\pi}$, then $|f(x_n) - f(y_n)| > C|x_n - y_n|$ if and only if

$$\frac{2}{\pi} \cdot \frac{(2n\pi + \frac{\pi}{2})(2n\pi)}{2n\pi + \frac{\pi}{2}} = 4n > C.$$

Therefore, for any $C > 0$, there are $x, y \in [0, 1]$ such that $|f(x) - f(y)| > C|x - y|$ and hence f is not a Lipschitz function on $[0, 1]$.

Proposition 4.10 *Let A be a non-empty closed subset of $\mathbb{R}$. If $f : A \to A$ is a contraction, then there is a fixed point of f, that is, there is a point $a \in A$ such that $f(a) = a$.*

Proof: Since f is a contraction on A, there is $0 < C < 1$ such that $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in A$. Fix $x_1 \in A$. Since $f(A) \subseteq A$, we can inductively define a sequence (x_n) in A by $x_{n+1} = f(x_n)$ for $n = 1, 2...$ Notice that we have

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| \leq C|x_n - x_{n-1}|$$

for all $n = 2, 3...$ This gives

$$|x_{n+1} - x_n| \leq C^{n-1}|x_2 - x_1|$$

for $n = 2, 3, ...$ So, for any $n, p = 1, 2..$, we see that

$$|x_{n+p} - x_n| \leq \sum_{i=n}^{n+p-1} |x_{i+1} - x_i| \leq |x_2 - x_1| \sum_{i=n}^{n+p-1} C^{i-1}.$$

Since $0 < C < 1$, for any $\varepsilon > 0$, there is N such that $\sum_{i=n}^{n+p-1} C^{i-1} < \varepsilon$ for all $n \geq N$ and $p = 1, 2, ...$ Therefore, (x_n) is a Cauchy sequence and thus the limit $a := \lim_n x_n$ exists. Since A is closed, we have $a \in A$ and hence f is continuous at a. On the other hand, since $x_{n+1} = f(x_n)$. Therefore, we have $a = f(a)$ by taking $n \to \infty$. The proof is finished. □

Remark 4.11 The Proposition 4.10 does not hold if f is not a contraction. For example, if we consider $f(x) = x - 1$ for $x \in \mathbb{R}$, then it is clear that $|f(x) - f(y)| = |x - y|$ and f has no fixed point in $\mathbb{R}$.

5 Continuous functions defined on intervals

Theorem 5.1 (Intermediate Value Theorem): *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose that $f(a) < z < f(b)$. Then there is c between a and b such that $f(c) = z$.*

Proof: Notice that if we consider the function $x \in [a, b] \mapsto f(x) - z$, then we may assume that $z = 0$.
Method I: Let

$$S := \{x \in [a, b] : f(x) \leq 0\}.$$

Notice that the set S is non-empty since $a \in S$ and is bounded. Then by the axiom of completeness, the supremum $c := \sup\{x \in S\}$ exists. Then $c \in [a, b]$ and there is a sequence in S such that $x_n \to c$. This, together with the continuity of f, imply that $f(c) = \lim_n f(x_n) \leq 0$ since $x_n \in S$. On the other hand, since $b \notin S$, we see that $c \in [a, b)$. Therefore, we can find a sequence (y_n) with $c < y_n < b$ for all n such that $y_n \to c+$ respectively. By using the continuity of f again, we see that $f(c) = \lim_n f(y_n) \geq 0$ because $y_n \notin S$. Therefore, $f(c) = 0$. The proof is finished.
Method II: Put $x_1 = a$ and $y_1 = b$. Now if $f(\frac{a+b}{2}) = 0$, then the result is obtained. If $f(\frac{a+b}{2}) > 0$, then we set $x_2 = a$ and $y_2 = \frac{a+b}{2}$. Similarly, if $f(\frac{a+b}{2}) < 0$, then we set $x_2 = \frac{a+b}{2}$ and $y_2 = b$. To repeat the same procedure, if there are x_N and y_N such that $f(\frac{x_N+y_N}{2}) = 0$, then the result is shown. Otherwise, we can find a decreasing sequence of closed and bounded intervals $[a, b] = [x_1, y_1] \supseteq [x_2, y_2] \supseteq \cdots$ with $\lim(y_n - x_n) = 0$ and $f(x_n) < 0 < f(y_n)$ for all n. Then by the Nested Intervals Theorem, we have $\bigcap_n [x_n, y_n] = \{c\}$ for some $c \in [x_1, y_1] = [a, b]$. Moreover, we have $\lim_n x_n = \lim_n y_n = c$. Then by the continuity of f, we see that $f(c) = \lim f(x_n) = \lim f(y_n)$. Since $f(x_n) < 0 < f(y_n)$ for all n, we have $f(c) = 0$. The proof is finished. $\square$

Remark 5.2 The assumption of the intervals in the Intermediate Value Theorem is essential. For example, consider $I = [0, 1) \cup (2, 3]$ and define $f : I \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \in [0, 1) \\ x - 1 & \text{if } x \in (2, 3]. \end{cases}$$

Then $f(0) < 1 < f(3)$ but $1 \notin f(I)$.

Recall that a non-empty subset I of $\mathbb{R}$ is called an interval if it has one of the following forms.

(i) $\mathbb{R}$.

(ii) $(-\infty, a]$ or $[a, \infty)$ or $(-\infty, a)$ or (a, ∞) for some $a \in \mathbb{R}$.

(iii) (a, b) or $(a, b]$ or $[a, b)$ or $[a, b]$ for some $a, b \in \mathbb{R}$ with $a < b$.

Lemma 5.3 *Let I be a non-empty subset of $\mathbb{R}$. Suppose that there are different elements in I. Then I is an interval if and only if for any $a, b \in I$ with $a < b$, we have $[a, b] \subseteq I$.*

Corollary 5.4 *Let $f : [a, b] \to \mathbb{R}$. Suppose that $M := \sup\{f(x) : x \in [a, b]\}$ and $m = \inf\{f(x) : x \in [a, b]\}$. Then $f([a, b]) = [m, M]$.*

Proof: Notice that if $m = M$, then f is a constant function and hence, the result is clearly true.
Now suppose that $m < M$. It is clear that $f([a, b]) \subseteq [m, M]$ because $m \le f(x) \le M$ for all $x \in [a, b]$. For the converse inclusion, notice that since $[a, b]$ is compact, there are x_1 and x_2 in $[a, b]$ such that $f(x_1) = m$ and $f(x_2) = M$. We may assume that $x_1 < x_2$. To apply the Intermediate Value Theorem for the restriction of f on $[x_1, x_2]$, we have $[m, M] \subseteq f([x_1, x_2]) \subseteq f([a, b])$. The proof is finished. □

Corollary 5.5 *Let I be an interval and let $f : I \to \mathbb{R}$ be a continuous non-constant function. Then $f(I)$ is an interval.*

Proof: Notice that by Lemma 5.3, it needs to show that for any $c, d \in f(I)$ with $c < d$ implies that $[c, d] \subseteq f(I)$. Suppose that $a, b \in I$ with $a < b$ satisfy $f(a) = c$ and $f(b) = d$. Notice that $[a, b] \subseteq I$ because I is an interval. If we put $M = \sup_{x\in[a,b]} f(x)$ and $m = \inf_{x\in[a,b]} f(x)$, then by Corollary 5.4, we have

$$[c, d] \subseteq [m, M] = f([a, b]) \subseteq f(I).$$

The proof is finished. □

Example 5.6 It is impossible to find a continuous surjection from (a, b) onto $(c, d) \cup (e, f)$ where $d \le e$.

6 Appendix: Open subsets of $\mathbb{R}$

Definition 6.1 Let V be a subset of $\mathbb{R}$.

(i) A point $c \in V$ is called an interior point of V if there is $r > 0$ such that $(c - r, c + r) \subseteq V$.

(ii) V is said to be an open subset of $\mathbb{R}$ is for every element in V is an interior point of V. In this case, if $x_0 \in V$, then V is called an open neighborhood of the point x_0.

Example 6.2 With the notation as above, we have

(i) All open intervals are open subsets of $\mathbb{R}$.

(ii) $\emptyset$ and $\mathbb{R}$ are open subsets.

(iii) Any closed and bounded interval is not an open subset.

(iv) The set of all rational numbers $\mathbb{Q}$ is neither open nor closed subset.

Proposition 6.3 *A non-empty subset A of $\mathbb{R}$ is open if and only if there is sequence of open intervals $I_n = (a_n, b_n)$ for $n = 1, 2, ...$ such that $A = \bigcup_{n=1}^{\infty} I_n$ and $I_n \cap I_m = \emptyset$ for $m \neq n$.*

Proof: Assume that A is an open subset. Notice that $\overline{\mathbb{Q}} = \mathbb{R}$. Since A is open, we see that $A \cap \mathbb{Q}$ is also a non-empty countable subset. Let $A \cap \mathbb{Q} = \{x_1, x_2, ...\}$. For each x_k, put $I_k := \bigcup\{J : x_k \in J \text{ and } J \text{ is an open interval}\}$. Then $X = \bigcup_{k=1}^{\infty} I_k$. On the other hand, we notice that I_k is also any open interval **(Why??)**. From this, we see that $I_k \cap I_j = \emptyset$ or $I_k = I_j$. Thus, we can find a subsequence (x_{n_k}) such that $I_{n_k} \cap I_{n_j} = \emptyset$ for $k \neq j$. Thus the sequence of disjoint open intervals $(I_{n_k})_{k=1}^{\infty}$ that we want.
The converse is clear. □

Recall that a point $c \in \mathbb{R}$ is called a *limit point (or cluster point)* of a subset A of $\mathbb{R}$ if for any $\delta > 0$, we have $(c-\delta, c+\delta) \cap A \neq \emptyset$.
Moreover, A is said to be a closed subset of $\mathbb{R}$ if A contains all its limit points. Let us recall the following useful fact that we have used many times.

Proposition 6.4 *Let A be a subset of $\mathbb{R}$. Then the following statements are equivalent.*

(i) A is closed.

(ii) If (x_n) is a sequence in A and $\lim x_n$ exists, then $\lim x_n \in A$.

The following is an important relation between the notion of openness and closeness.

Proposition 6.5 *A subset A of $\mathbb{R}$ is open if and only if its complement $A^c = \mathbb{R} \setminus A$ is closed in $\mathbb{R}$.*

Proof: For $(\Rightarrow)$, we suppose that A is open first but A^c is not closed. Then there is a limit point c of A^c but $c \notin A^c$ and hence, $c \in A$. This implies that there is $r > 0$ such that $(c-r, c+r) \subseteq A$ because A is open and thus, $(c-r, c+r) \cap A^c = \emptyset$. It contradicts to the assumption of c being a limit point of A^c.
For the converse, assume that A is not an open subset. Then there is a point $c \in A$ which is not an interior. Thus, for any $r > 0$, we have $(c-r, c+r) \subsetneq A$. For considering $r = 1/n$, we can find a sequence in (x_n) in A^c such that $\lim x_n = c$. Notice that $x_n \neq c$ for all n because $c \notin A^c$. This implies that c is a limit point of A^c but $c \notin A^c$ and thus, A^c is not closed. The proof is finished. □

Next, let us recall a very important concept in mathematics. A function f is said to be continuous on a subset A of $\mathbb{R}$ if every point $c \in A$ and for any $\varepsilon > 0$, there is $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon \text{ whenever } |x - c| < \delta \text{ and } x \in A.$$

This is equivalent to saying that

$$(c-\delta, c+\delta) \cap A \subseteq f^{-1}\big((f(c)-\varepsilon, f(c)+\varepsilon)\big) \tag{6.1}$$

The following is an important characterization of a continuous map which can be generalized to the case of a general topological space.

Strongly recommend: Take the courses for the next: Mathematical III, Real Analysis, Complex Variables with Applications and Introduction to Topology.

Proposition 6.6 *Let $f : A \to \mathbb{R}$ be a function defined on a subset A of $\mathbb{R}$. Then f is continuous on A if and only if for any open subset W of $\mathbb{R}$, there is an open subset V of $\mathbb{R}$ such that $V \cap A = f^{-1}(W)$.*

Proof: Assume that f is continuous on A. Let W be any open subset of $\mathbb{R}$. If $f^{-1}(W) = \emptyset$, then we just simply take $V = \emptyset$ as required. Now it suffices to consider the case of $f^{-1}(W) \neq \emptyset$. Note that if $c \in f^{-1}(W) \subseteq A$, then there is $\varepsilon_c > 0$ such that $(f(c) - \varepsilon_c, f(c) + \varepsilon_c) \subseteq W$ because W is open. By using Equation 6.1, we can find $\delta_c > 0$ such that

$$(c - \delta_c, c + \delta_c) \cap A \subseteq f^{-1}\big((f(c) - \varepsilon_c, f(c) + \varepsilon_c)\big) \subseteq f^{-1}(W).$$

If we let $V := \bigcup_{c \in f^{-1}(W)} (c - \delta_c, c + \delta_c)$, then V is open and $V \cap A = f^{-1}(W)$ as desired.
Conversely, let $c \in A$, we are going to show that f is continuous at c. Let $\varepsilon > 0$. Then by the assumption, there is an open set V such that $V \cap A = f^{-1}(W)$, where $W := (f(c) - \varepsilon, f(c) + \varepsilon)$. Since V is open and $c \in f^{-1}(W) = c \in V \cap A$, there is $\delta > 0$ such that $(c - \delta, c + \delta) \subseteq V$ and thus, we have $|f(x) - f(c)| < \varepsilon$ as $x \in A$ and $|x - c| < \delta$. Therefore, f is continuous at c. The proof is finished. □

Definition 6.7 A subset A of $\mathbb{R}$ is said to be disconnected if there are a pair of open subsets U and V of $\mathbb{R}$ with $A \subseteq U \cup V$ such that $U \cap A$ and $V \cap A$ both are non-empty but $(U \cap A) \cap (V \cap A) = \emptyset$.
If A is not disconnected, then A is said to be connected.

Proposition 6.8 *Let A be a subset of $\mathbb{R}$. Suppose that A contains at least two elements. Then A is connected if and only if A is an interval.*

Proof: The result is equivalent to saying that A is disconnected if and only if A is not an interval. Suppose that A is not an interval. Then by using Lemma 5.3, there are $a, b \in A$ such that $[a, b] \nsubseteq A$. Let $c \in [a, b] \setminus A$. Notice that $a < c < b$ since $a, b \in A$. Put $U := (-\infty, c)$ and $V := (c, \infty)$. Then the pair of open sets U and V satisfy the condition in Definition 6.7 as above, and thus, A is disconnected.
Now suppose that A is a disconnected set but A is an interval. Let U and V be the open sets as in Definition 6.7. Then we can find some points $a \in U \cap A$ and $b \in V \cap A$. We may assume that $a < b$. Notice that since U is open, we see that the set $S := \{u_1 \in (a, b) : [a, u_1] \subseteq U\}$ is a non-empty bounded set and thus, one can define $u := \sup S$. On the other hand, since A is an interval by the assumption, we have $u \in [a, b] \subseteq A \subseteq U \cup V$. Since U is open, if $u \in U$, then we can find some $w \in (u, b)$ such that $[u, w] \subseteq U$ which contradicts to u being the supremum of the above set S.
On the other hand, if $u \in V$, then there is $\delta > 0$ such that $u - \delta < u_1 \leq u$ for some $u_1 \in S$ and $(u - \delta, u) \subseteq V$ by the definition of supremum and V is open. This implies that $u_1 \in (U \cap A) \cap (V \cap A)$ that contradicts to the fact that $(U \cap A) \cap (V \cap A)$ is empty. Therefore, A must not be an interval. The proof is finished. □

Remark 6.9 In Proposition 6.8, we have shown that for a subset of $\mathbb{R}$, there is no different between a connected set and an interval. Also, at a first glimpse of Definition 6.7, it seems that the definition of a connected set is more complicated than the definition of an interval. It is quite natural to ask why we have to introduce the connectedness of a set. In fact, the definition of an interval is given by the order structure of $\mathbb{R}$. Notice that Definition 6.7 is defined by the distance structure *(more precise, the topological structure)* of $\mathbb{R}$. Therefore, Proposition 6.8 tells us that Definition 6.7 is a suitable generalization of the concept of "interval" in the case of a general topological space.

We are going to give another proof of the *Intermediate Value Theorem.*

Theorem 6.10 (Intermediate Value Theorem): *If f is a continuous non-constant function defined on an interval D, then $f(D)$ is an interval.*

Proof: By using Proposition 6.8, the Theorem is equivalent to saying that $f(D)$ is connected if D is connected, that is, the connectedness of a set is preserved under a continuous map. Suppose that $f(D)$ is disconnected. As in Definition 6.7, let U and V be the pair of open subsets such that $f(D) \subseteq U \cap V$ with $f(D) \cap U$ and $f(D) \cap V$ being non-empty and $(f(D) \cap U) \cap (f(D) \cap V) = \emptyset$. Then by Proposition 6.6, we can find a pair open subsets E and F such that $E \cap D = f^{-1}(U)$ and $F \cap D = f^{-1}(W)$. Then the sets E and F satisfy the condition in Definition 6.7 for the domain D and thus, D is disconnected. The proof is finished. □

References

[1] R.G. Bartle and I.D. Sherbert, Introduction to Real Analysis, ($4th$ ed), Wiley, (2011).